Max-Planck-Institut für Mathematik
Bonn

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Preprint Series 2020 (13)

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MPIM 20-13

AN ALGEBRAIC-GEOMETRIC CONSTRUCTION OF IND-VARIETIES OF GENERALIZED FLAGS

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Abstract. We define the class of admissible linear embeddings of flag varieties. The definition is given in the general language of algebraic geometry. We then prove that an admissible linear embedding of flag varieties has a certain explicit form in terms of linear algebra. This result enables us to show that any direct limit of admissible embeddings of flag varieties is isomorphic to an ind-variety of generalized flags as defined in [1]. These latter ind-varieties have been introduced in terms of the ind-group $SL(\infty)$ (respectively, $O(\infty)$ or $Sp(\infty)$ for isotropic generalized flags), and the current paper constructs them in purely algebraic-geometric terms.

2010 Mathematics Subject Classification: Primary 14M15; Secondary 14J60, 32L05.
Bibliography: 4 items.
Keywords: flag variety, linear embedding of flag varieties, homogeneous ind-variety, generalized flag.

1. Introduction

Flag varieties play a fundamental role in geometry, and so do their analogues in ind-geometry. In this paper, we would like to place these analogues under the looking glass and provide a new characterization of the ind-varieties of generalized flags constructed in [1]. Around 20 years ago, I. Dimitrov and the first author realized that in the context of ind-geometry the notion of a flag of vector subspaces in an ambient infinite-dimensional vector space is rather subtle. More precisely, in addition to the obvious three types of infinite flags, that is, chains of vector subspaces enumerated by $\mathbb{Z}_{>0}$, $\mathbb{Z}_{<0}$ or $\mathbb{Z}$, there is the need to consider chains of subspaces enumerated by more general totally ordered sets in which every element has an immediate predecessor or an immediate successor, but possibly not both. Such chains, satisfying the additional condition that every vector of the ambient vector space is contained in some space of the chain but not in its immediate predecessor, were christened *generalized flags* in [1]. The main result of [1] can be summarized roughly as follows: generalized flags in a countable-dimensional vector space are in a natural 1-1 correspondence with splitting parabolic subgroups P of the ind-group $GL(\infty)$, and hence the points of homogeneous ind-spaces of the form $GL(\infty)/P$ can be thought of as generalized flags. A similar statement about isotropic generalized flags holds for the ind-groups $O(\infty)$ and $Sp(\infty)$. In particular, the concept of generalized flag, and therefore also the notion of an ind-variety of generalized flags, has been motivated in the past by the notion of a parabolic subgroup of an ind-group like $GL(\infty)$, $O(\infty)$, $Sp(\infty)$.

The main purpose of the present paper is to propose another, purely algebraic-geometric, approach to the ind-varieties of generalized flags. More precisely, we define *admissible linear embeddings* of usual flag varieties

$$Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V') \tag{1}$$

and show that an ind-variety obtained as a direct limit of such linear embeddings is isomorphic to an ind-variety of generalized flags. In particular, such a linear direct limit is automatically a homogeneous ind-space of $GL(\infty)$. We also consider isotropic generalized flags and prove a similar result for the ind-groups $O(\infty)$ and $Sp(\infty)$. In this way, the notion of an admissible linear embedding of flag varieties leads naturally to the concept of generalized flag. A small part of this program has already been carried in our paper [3] where we characterize linear embeddings of grassmannians, and then as a consequence describe linear ind-grassmannians up to isomorphism.

Our main new result concerning embeddings of finite-dimensional flag varieties is finding an explicit form of a class of embeddings (1) which we call *admissible*. We define an admissible linear embedding in general algebraic-geometric terms, and then show that such an embedding is nothing but an extension of a flag from $Fl(m_1, ..., m_k, V)$ to a possibly longer flag in $Fl(n_1, ..., n_{\tilde{k}}, V')$, given by an explicit formula from linear algebra. We call the latter embeddings *standard extensions*. This enables us to prove that a direct limit of admissible linear embeddings is isomorphic to an ind-variety of generalized flags as in [1], as it is relatively straightforward to show that direct limits of standard extensions have this property.

The paper is concluded by an appendix in which we present two examples of direct limits of linear but non-admissible embeddings of flag varieties, that are not isomorphic to ind-varieties of generalized flags.

Acknowledgements. I.P. thanks Vera Serganova for a useful discussion, which took place several years ago, on the general idea of an algebraic-geometric approach to ind-varieties of generalized flags. I.P. was supported in part by DFG-grant PE 980/7-1. A.S.T worked on this article within the framework of the Academic Fund Program at HSE University in 2020-2021 (grant number 20-01-011) and within the framework of the Russian Academic Excellence Project "5-100". A.S.T. thanks the Max Planck Institute for Mathematics in Bonn, where this work was partially done during the winter of 2020, for hospitality and financial support.

Notation. The sign $\subset$ stands for not necessarily strict set-theoretic inclusion. By $G(m, V)$ we denote the grassmannian of m-dimensional subspaces of V for $1 \leq m \leq \dim V$. We also use the notation $\mathbb{P}(V)$ for $G(1, V)$. If $a : X \to Y$ is a morphism of algebraic varieties, by a^* and a_* we denote respectively the pullback or pushforward of vector bundles. The superscript $(\cdot)^\vee$ indicates dual space or dual vector bundle.

2. Definition of linear embedding of flag varieties

In this section we give the basic definitions of linear embeddings of flag varieties including the case of isotropic flag varieties.

The base field is $\mathbb{C}$ and all vector spaces, varieties and ind-varieties considered below are defined over $\mathbb{C}$. Let V be a vector space of dimension $\dim V \geq 2$. For any increasing sequence of positive integers $1 \leq m_1 < ... < m_k < \dim V$, we consider the *flag variety* $Fl(m_1, ..., m_k, V) := \{(V_{m_1}, ..., V_{m_k}) \in G(m_1, V) \times ... \times G(m_k, V) \mid V_{m_1} \subset ... \subset V_{m_k}\}$. We denote its points by $F = (0 \subset V_{m_1} \subset ... \subset V_{m_k} \subset V)$ or sometimes by $F = (V_{m_1} \subset ... \subset V_{m_k})$. The ordered k-tuple $(m_1, ..., m_k)$ is the *type* of a flag $F \in Fl(m_1, ..., m_k, V)$.

There is a natural embedding

$$j : \ Fl(m_1, ..., m_k, V) \hookrightarrow G(m_1, V) \times ... \times G(m_k, V)$$

and there are projections

$$\pi_i : \ Fl(m_1, ..., m_k, V) \to G(m_i, V), \ F = (V_{m_1} \subset ... \subset V_{m_k}) \mapsto V_{m_i}, \ i = 1, ..., k.$$

We have

$$\text{Pic } Fl(m_1, ..., m_k, V) = \mathbb{Z}[L_1] \oplus ... \oplus \mathbb{Z}[L_k],$$

where

$$L_i := \pi_i^* \mathcal{O}_{G(m_i,V)}(1), \quad i = 1, ..., k.$$

Here, $\mathcal{O}_{G(m_i,V)}(1)$ denotes the invertible sheaf on $G(m_i, V)$ satisfying $H^0(\mathcal{O}_{G(m_i,V)}(1)) = \wedge^{m_i}(V^*)$. By definition, $[L_1], ..., [L_k]$ is a *preferred set of generators* of Pic $Fl(m_1, ..., m_k, V)$.

Let V be equipped with a non-degenerate symmetric bilinear form on V. For our purposes, we can assume that $\dim V \geq 7$. For $1 \leq k \leq [\frac{\dim V}{2}]$, the *orthogonal grassmannian* $GO(m, V)$ is defined as the subvariety of $G(m, V)$ consisting of isotropic m-dimensional subspaces of V. Unless $\dim V = 2m$, the variety $GO(m, V)$ is a smooth irreducible variety. For $\dim V = 2m$, the orthogonal grassmannian is a disjoint union of two isomorphic smooth irreducible components, and they are both isomorphic to $GO(m-1, V')$ where $\dim V' = 2m-1$. Slightly abusing notation, we will denote by $GO(m, V)$ each of these two components.

If $m \neq \frac{\dim V}{2} - 1$, then $\operatorname{Pic} GO(m, V) = \mathbb{Z}[\mathcal{O}_{GO(m,V)}(1)]$, where the sheaf $\mathcal{O}_{GO(m,V)}(1)$ posesses the following property: if $t : GO(m, V) \hookrightarrow G(m, V)$ is the tautological embedding, then

$$t^*\mathcal{O}_{G(m,V)}(1) \cong \begin{cases} \mathcal{O}_{GO(m,V)}(1) & \text{for} \quad m \neq \frac{\dim V}{2}, \\ \mathcal{O}_{GO(k,V)}(2) & \text{for} \quad m = \frac{\dim V}{2}. \end{cases}$$

If $m = \frac{\dim V}{2} - 1$, then for any $V_{m-1} \in GO(m-1, V)$ there is a unique $V_m \in GO(m, V)$ such that $V_{m-1} \subset V_m$. Thus there is a well-defined morphism

$$\theta : \ GO(m-1, V) \to GO(m, V), \quad V_{m-1} \mapsto V_m, \quad \text{where} \quad V_m \supset V_{m-1}. \tag{2}$$

Consequently,

$$\operatorname{Pic} GO(m-1, V) = \mathbb{Z}[\theta^*\mathcal{O}_{GO(m,V)}(1)] \oplus \mathbb{Z}[\mathcal{O}_{GO(m-1,V)}(1)],$$

where by $\mathcal{O}_{GO(m-1,V)}(1)$ we denote the θ-relatively ample Grothendieck sheaf determined by the property that $\theta_*\mathcal{O}_{GO(m-1,V)}(1)$ is the universal quotient bundle on $GO(m, V)$.

Next, let $1 \leq m_1 < ... < m_k$ be an increasing sequence of positive integers, where $m_k \leq [\frac{\dim V}{2}]$. The *orthogonal flag variety* $FlO(m_1, ..., m_k, V)$ is defined as

$$FlO(m_1, ..., m_k, V) := \{(V_{m_1}, ..., V_{m_k}) \mid V_{m_1} \in GO(m_i, V), \ V_{m_1} \subset ... \subset V_{m_k}\},$$

where, according to our convention, we assume $GO(m_k, V)$ connected if $m_k = \frac{\dim V}{2}$. Similarly to the case of usual flag varieties, there is a natural embedding $j : FlO(m_1, ..., m_k, V) \hookrightarrow GO(m_1, V) \times ... \times GO(m_k, V)$ and there are projections $\pi_i : Fl(m_1, ..., m_k, V) \to GO(m_i, V)$, $(V_{m_1} \subset ... \subset V_{m_k}) \mapsto V_{m_i}$, $i = 1, ..., k$. Unless $m_k = \frac{\dim V}{2} - 1$, we have

$$\operatorname{Pic} FlO(m_1, ..., m_k, V) = \mathbb{Z}[L_1] \oplus ... \oplus \mathbb{Z}[L_k],$$

where

$$L_i := \pi_i^*\mathcal{O}_{GO(m_i,V)}(1), \quad i = 1, ..., k.$$

The isomorphism classes $[L_i]$ are a *preferred set of generators* of $\operatorname{Pic} FlO(m_1, ..., m_k, V)$. If $m_k = \frac{\dim V}{2} - 1$, then there is an additional preferred generator $[(\theta \circ \pi_{k-1})^*\mathcal{O}_{GO(m_k+1,V)}(1)]$ of $\operatorname{Pic} FlO(m_1, ..., m_k, V)$.

Let now V be equipped with a non-degenerate symplectic form. This implies that $\dim V \in 2\mathbb{Z}_{>0}$. Assume $1 \leq m \leq \frac{1}{2}\dim V$. By definition, the *$m$-th symplectic grassmannian* $GS(m, V)$ is the smooth irreducible subvariety of $G(m, V)$ consisting of isotropic m-dimensional subspaces of V. It is known that

$$\operatorname{Pic} GS(m, V) = \mathbb{Z}[\mathcal{O}_{GS(k,V)}(1)], \quad \mathcal{O}_{GS(k,V)}(1) = i^*\mathcal{O}_{G(k,V)}(1),$$

where $i : GS(m, V) \hookrightarrow G(m, V)$ is the tautological embedding. For a fixed increasing sequence of positive integers $1 \leq m_1 < ... \leq m_k \leq \frac{\dim V}{2}$, the *symplectic flag variety* is defined as

$$FlS(m_1, ..., m_k, V) := \{(V_{m_1}, ..., V_{m_k}) \in GS(m_1, V) \times ... \times GS(m_k, V) \mid V_{m_1} \subset ... \subset V_{m_k}\}.$$

We have a natural embedding $j : FlS(m_1, ..., m_k, V) \hookrightarrow GS(m_1, V) \times ... \times GS(m_k, V)$ and projections $\pi_i : Fl(m_1, ..., m_k, V) \to GS(m_i, V)$, $(V_{m_1} \subset ... \subset V_{m_k}) \mapsto V_{m_i}$, $i = 1, ..., k,$. Moreover,

$$\operatorname{Pic} FlS(m_1, ..., m_k, V) = \mathbb{Z}[L_1] \oplus ... \oplus \mathbb{Z}[L_k],$$

where

$$L_i := \pi_i^* \mathcal{O}_{GS(m_i,V)}(1), \quad i = 1, ..., k,$$

The isomorphism classes $[L_i]$ are a *preferred set of generators* of Pic $FlS(m_1, ..., m_k, V)$.

We now proceed to the definition of linear embeddings of flag varieties and their orthogonal and symplectic analogues.

Definition 2.1. Let k and $\tilde{k}$ be positive integers with $1 < k \leq \tilde{k}$. An embedding of flag varieties

$$\varphi : \ X \hookrightarrow Y,$$

where $X = Fl(m_1, ..., m_k, V)$, $Y = Fl(n_1, ..., n_{\tilde{k}}, V')$, or $X = FlO(m_1, ..., m_k, V)$, $Y = FlO(n_1, ..., n_{\tilde{k}}, V')$, or $X = FlS(m_1, ..., m_k, V)$, $Y = FlS(n_1, ..., n_{\tilde{k}}, V')$, is a *linear embedding* if, for any j, $1 \leq j \leq \tilde{k}$, we have

$$[\varphi^* M_j] = 0 \quad \text{or} \quad [\varphi^* M_j] = [L_i]$$

for some i, $1 \leq i \leq k$, where $[L_1], ..., [L_k]$ and $[M_1], ..., [M_{\tilde{k}}]$ are the preferred sets of generators of Pic$X$ and Pic$Y$.

Example 2.2. Assume that $k = \tilde{k} = 1$ in Definition 2.1. Then X and Y are grassmannians, orthogonal grassmannians, or symplectic grassmannians. In all cases, except when $X = GO(m, V)$ and $Y = GO(n, V')$ for $(m, \dim V) = (l-1, 2l)$ or $(n, \dim V') = (r-1, 2r)$, a linear embedding $\varphi : X \to X$ is simply an embedding with $\varphi^*[M] = [L]$, where $[L]$ and $[M]$ are respective ample generators of the Picard groups Pic$Y$ and Pic$X$, cf. [3, Def. 2.1].

In the remaining cases, a linear embedding $\varphi : X \to Y$ exists if and only if $X \simeq GO(l-1, V)$, $Y \simeq GO(r-1, V')$ for $l \leq r$, and here the linearity of φ implies $\varphi^* \mathcal{O}_{GO(r-1,V')}(1) \cong \mathcal{O}_{GO(l-1,V)}(1)$, $\varphi^* \theta'^* \mathcal{O}_{GO(r,V')}(1) \cong \theta^* \mathcal{O}_{GO(l,V)}(1)$, where $\theta : GO(l-1, V) \to GO(l, V)$ and $\theta' : GO(r-1, V') \to GO(r, V')$ are the projections defined in (2). To see this, one has to show (we leave this to the reader) that it is impossible to have an embedding $\varphi : \ GO(l-1, V) \to GO(r-1, V')$ with $\varphi^* \theta'^* \mathcal{O}_{GO(r,V')}(1) \cong \mathcal{O}_{GO(l-1,V)}(1)$, $\varphi^* \mathcal{O}_{GO(r-1,V')}(1) \cong \theta^* \mathcal{O}_{GO(l,V)}(1)$.

A linear embedding φ as in Definition 2.1 induces a partition with $k+1$ parts $\{0, 1, ..., \tilde{k}, \tilde{k}+1\} = I_0 \sqcup I_1 \sqcup I_2 \sqcup ... \sqcup I_k$ such that $0 \in I_0$ and $j \in I_0$ iff $\varphi^*[M_j] = 0$, respectively, $j \in I_i$ for $i \geq 1$ iff $\varphi^*[M_j] = [L_i]$. The map $j \mapsto i$, for $j \in I_i$, is a surjection which we denote by p. By definition, $p(0) = 0$.

Proposition 2.3. *(i) Let $\varphi : \ Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$ be a linear embedding. Then φ induces a collection of morphisms of grassmannians*

$$\varphi_{[i]} = \{\varphi_{i,j}\}_{i=p(j)} : \ G(m_i, V) \to \prod_{j>0:p(j)=i} G(n_j, V'), \quad 0 \leq i \leq k,$$

such that the diagram

$$(3) \qquad \begin{array}{ccc} Fl(m_1, ..., m_k, V) & \xhookrightarrow{\varphi} & Fl(n_1, ..., n_{\tilde{k}}, V') \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ G_0 \times G(m_1, V) \times ... \times G(m_k, V) & \xhookrightarrow{\varphi_{[1]} \times ... \times \varphi_{[k]}} & G(n_1, V') \times ... \times G(n_{\tilde{k}}, V') \end{array}$$

where j and j' are the natural embeddings, is commutative. Here G_0 is a single point, and is present in the diagram if and only if there are constant morphisms $\varphi_{0=p(j),j} : G_0 \to G(n_j, V')$.
(ii) Similar statements hold in the orthogonal and symplectic cases.

In the proof, we will need the following.

Lemma 2.4. *Let X, Y, Z be projective varieties with Y smooth, and let $a : X \to Y$ and $b : X \to Z$ be morphisms such that a is surjective and b is constant on the fibers of a. Then there exists a morphism $f : Y \to Z$ such that $b = f \circ a$.*

Proof. Consider the morphism $g : X \to Y \times Z,\ x \mapsto (a(x), b(x))$, and let $Y \stackrel{a'}{\leftarrow} Y \times Z \stackrel{b'}{\to} Z$ be the projections onto factors so that $a = a' \circ g$ and $b = b' \circ g$. Since b is constant on the fibers of p, it follows that $\tilde{a} := a'|_{g(X)} : g(X) \to Y$ is a bijection. Therefore, as Y is smooth, $\tilde{a}$ is an isomorphism (see, e.g., [4, Ch.2, Section 4.4, Thm. 2.16]). The desired morphism f is now the composition $f = b' \circ \tilde{a}^{-1}$. □

Proof of Proposition 2.3. (i) We consider the case $k = \tilde{k} = 2$. For arbitrary $k,\ \tilde{k}$ the proof goes along the same lines, and we leave the details to the reader. Set $[L_1] := \varphi^*[M_{j_1}],\ [L_2] := \varphi^*[M_{j_2}]$, and let $\pi_i : Fl(m_1, m_2, V) \to G(m_i, V),\ \pi'_i : Fl(n_1, n_2, V') \to G(n_i, V'),\ i = 1, 2$, be the natural projections. For an arbitrary point $x = (x_1, x_2) = (V_{m_1}, V_{m_2}) \in Fl(m_1, m_2, V) \subset G(m_1, V) \times G(m_2, V)$, consider the fibres $\pi_i^{-1}(x_i) \subset F,\ i = 1, 2$, through the point x. Since φ is a linear embedding, we have $M_{j_1}|_{\varphi(\pi_1^{-1}(x_1))} \simeq \varphi^* M_{j_1}|_{\pi_1^{-1}(x_1)} \simeq \mathcal{O}_{\pi_1^{-1}(x_1)} \simeq \mathcal{O}_{\varphi(\pi_1^{-1}(x_1))}$. As $\varphi(\pi_1^{-1}(x_1))$ is an irreducible variety and $M_{j_1} = \pi'^*_1\mathcal{O}_{G(n_{j_1}, V')}(1)$, where $\mathcal{O}_{G(n_{j_1}, V')}(1)$ is an ample sheaf, it follows from the above isomorphisms that π'_{j_1} is constant on the variety $\varphi(\pi_1^{-1}(x_1))$. Equivalently, the morphism $\pi'_{j_1} \circ \varphi$ is constant on the fibres of the projection π_1.

Lemma 2.4 implies that $\pi'_1 \circ \varphi$ factors through the projection π_1, i.e. there is a well-defined morphism

$$\varphi_1 :\ G(m_1, V) \to G(n_{j_1}, V'),\ x_1 \mapsto \pi'_{j_1}(\varphi(\pi_1^{-1}(x_1))) \tag{4}$$

such that $\varphi_1 \circ \pi_1 = \pi'_{j_1} \circ \varphi$. In a similar way there is a well-defined morphism

$$\varphi_2 :\ G(m_2, V) \to G(n_{j_2}, V'),\ x_2 \mapsto \pi'_{j_2}(\varphi(\pi_1^{-1}(x_2))) \tag{5}$$

such that $\varphi_2 \circ p_2 = \pi'_{j_2} \circ \varphi$. By construction, φ_1 and φ_2 are linear morphisms.

Considering now $Fl(m_1, m_2, V)$ and $Fl(n_1, n_2, V')$ as lying, respectively, in $G(m_1, V) \times G(m_2, V)$ and in $G(n_1, V') \times G(n_2, V')$, for any points $x = (x_1, x_2) \in Fl(m_1, m_2, V)$ and $x' = (x'_1, x'_2) \in Fl(n_1, n_2, V')$ we have

$$x = \pi_1^{-1}(x_1) \cap \pi_2^{-1}(x_2), \quad x' = {\pi'}_1^{-1}(x'_1) \cap {\pi'}_2^{-1}(x'_2).$$

This together with (4) and (5) shows that, if $x'_{j_i} = \varphi_i(x_i),\ i = 1, 2$, then

$$\varphi(x) = \varphi(\pi_1^{-1}(x_1)) \cap \varphi(\pi_2^{-1}(x_2)) \in {\pi'}_{j_1}^{-1}(x'_{j_1}) \cap {\pi'}_{j_2}^{-1}(x'_{j_2}) = (\varphi_1 \times \varphi_2)(x),$$

i.e. the diagram (3) is commutative for $k = 2$.

We leave to the reader to make (ii) precise and check that the above proof extends to this case. □

3. Standard extensions of flag varieties

In this section we introduce and study a class of embeddings of flag varieties that we call standard extensions. In almost all cases, standard extensions are linear embeddings in the sense of Section 2.

We start by considering the case of grassmannians. Let

$$\varphi : G(m, V) \hookrightarrow G(n, V') \tag{6}$$

be a regular morphism. Assume $\dim V' > \dim V,\ m \neq 0,\ m \neq \dim V$. We say that φ is a *strict standard extension* if there exists an isomorphism of vector spaces $V' = V \oplus \widehat{W}$ and a subspace $W \subset \widehat{W}$, such that

$$\varphi(V_m) = V_m \oplus W$$

where $V_m \subset V$ is an arbitrary point of $G(m,V)$. If $m = 0$ or $m = \dim V$, a morphism (6) is necessarily constant and we call it a *constant strict standard extension.* In this case we set $W := \varphi(G(m,V))$.

It is easy to check that a nonconstant strict standard extension is a linear embedding.

By a *modified standard extension* we understand an embedding (6) for which there exists a strict standard extension

$$\varphi' : G(m,V) \hookrightarrow G(\dim V' - n, V'^{\vee})$$

such that $\varphi = d \circ \varphi'$ where

$$d : G(\dim V' - n, V'^{\vee}) \xrightarrow{\sim} G(n, V')$$

is the duality isomorphism. In what follows, a *standard extension* will mean a strict standard extension or a modified standard extension.

Note that if a morphism (6) is linear, it is not necessarily a standard extension. For instance, the reader can prove that the Plücker embedding

$$\psi : \ G(m,V) \hookrightarrow G(1, \wedge^m V) = \mathbb{P}(\wedge^m V)$$

is a standard extension if and only if $m = 1$ or $m = \dim V - 1$. On the other hand, the Plücker embedding is of course a linear embedding.

In the case of orthogonal and symplectic grassmannians, a strict standard extension is defined in the same way with the additional requirement that the decomposition $V' = V \oplus U$ be orthogonal and that the spaces V_m and W are isotropic. In these cases there is no need to consider modified standard extensions (as the spaces V and $V^{\vee}$ are identified via the respective non-degenerate form), and the terms strict standard extension and standard extension are synonyms.

Here is a definition of strict standard extension φ of grassmannians which refers only to the data of linear algebra which can be recovered canonically from the embedding φ.

Definition 3.1. Let $\dim V' > \dim V$. A morphism of grassmannians $\varphi : G(m,V) \hookrightarrow G(n,V')$ is said to be a *strict standard extension* if either $G(m,V)$ is a point (i.e. $m = 0$ or $m = \dim V$, and φ is constant) or there exists a subspace $U \subset V'$ and a surjective linear operator $\varepsilon : \ U \twoheadrightarrow V$ such that

$$\varphi(V_m) = \varepsilon^{-1}(V_m). \tag{7}$$

If φ is a nonconstant standard extension, the subspace $U \subset V'$ is unique and the linear operator $\varepsilon : U \to V$ is unique up to a scalar multiple. Indeed, assume φ is given and set

$$W := \bigcap_{V_m \subset V} \varphi(V_m). \tag{8}$$

Let S and S' denote respectively the tautological bundles on $G(m,V)$ and $G(n,V')$. There is an obvious exact sequence

$$0 \to W \otimes \mathcal{O}_{G(m,V)} \to \varphi^* S' \to S \to 0.$$

Dualization yields an injective homomorphism $V^{\vee} = H^0(G(m,V), S^{\vee}) \hookrightarrow H^0(G(m,V), (\varphi^* S')^{\vee})$ with cokernel equal $W^{\vee}$. Set $U^{\vee} = H^0(G(m,V), (\varphi^* S')^{\vee})$. Then a second dualization yields a surjective homomorphism $\varepsilon : U \to V$ with $\ker \varepsilon = W$. In particular,

$$U = \bigcup_{V_m \subset V} \varphi(V_m). \tag{9}$$

In what follows, we will assign a subspace $U \subset V'$ also in the case when φ is constant: we set $U = W := \varphi(G(m,V)) \in G(n,V')$ and $\varepsilon = 0$. Formulas (7) and (9) then hold in this case too.

It is easy to show that Definition 3.1 is equivalent to the above "naive" definition of strict standard extension. Let φ be a nonconstant strict standard extension according to Definition 3.1. Then U and $\varepsilon : U \to V$ are given, and we can choose a splitting $U \simeq V \oplus (W = \ker \varepsilon)$. In particular, this induces an embedding V into V'. We then extend the splitting $U \simeq V \oplus W$ to a splitting $V' = V \oplus \widehat{W}$ where $W \subset \widehat{W}$. This yields the datum of "naive" definition. Conversely, given a nonconstant strict standard extension as in the "naive" definition, we simply set $U := V \oplus W$ and define ε to be the projection $U \to V$. Finally, if φ is constant then we put $U := \varphi(G(m,V)) = W$ (here $\dim U = n$).

In the orthogonal and symplectic cases, in Definition 3.1 one must assume that the space W is isotropic and the isomorphism $U/W \xrightarrow{\sim} V$ induced by the operator $\varepsilon : U \twoheadrightarrow V$ is an isomorphism of spaces endowed with symmetric, or respectively symplectic, forms. Here the form on U is induced by the respective form on V'.

It is a straightforward observation that in all cases the composition of standard extensions of grassmannians is also a standard extension. The composition of two strict standard extensions or two modified standard extensions is a strict standard extension, while the composition of a strict standard extension and a modified standard extension is again a modified standard extension.

We now give the definition of a strict standard extension of usual and isotropic flag varieties.

Definition 3.2. An embedding of flag varieties $\varphi : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$, respectively, $\varphi : FlO(m_1, ..., m_k, V) \hookrightarrow FlO(n_1, ..., n_{\tilde{k}}, V')$, respectively, $\varphi : FlS(m_1, ..., m_k, V) \hookrightarrow FlS(n_1, ..., n_{\tilde{k}}, V')$, is said to be a *strict standard extension*, or simply a *standard extension* in the orthogonal and symplectic cases, if there exists a flag of distinct nonzero subspaces of V',

$$U_1 \subset U_2 \subset ... \subset U_{\tilde{k}}$$

such that in the orthogonal and symplectic cases the spaces U_i are nondegenerate, and a commutative diagram

$$\begin{array}{ccccccc} V & = & V & = & \cdots & = & V \\ \uparrow{\scriptstyle \varepsilon_1} & & \uparrow{\scriptstyle \varepsilon_2} & & & & \uparrow{\scriptstyle \varepsilon_{\tilde{k}}} \\ U_1 & \hookrightarrow & U_2 & \hookrightarrow & \cdots & \hookrightarrow & U_{\tilde{k}} \end{array} \tag{10}$$

of linear operators $\varepsilon_i : U_i \to V$, surjective whenever nonzero, compatible with the respective forms on U_i and V and having isotropic kernels in the orthogonal and symplectic cases, and such that

$$\begin{aligned} &\varphi\big(0 = V_{\overline{p}(0)} \subset V_{\overline{p}(1)} \subset ... \subset V_{\overline{p}(\tilde{k})} \subset V_{\overline{p}(\tilde{k}+1)} = V\big) = \\ &\big(0 \subset \varepsilon_1^{-1}(V_{\overline{p}(1)}) \subset \varepsilon_1^{-1}(V_{\overline{p}(2)}) \subset ... \subset \varepsilon_{\tilde{k}}^{-1}(V_{\overline{p}(\tilde{k})}) \subset V'\big) \end{aligned} \tag{11}$$

for a suitable surjective map $\overline{p} : \{0, 1, ..., \tilde{k}, \tilde{k}+1\} \to \{0, 1, ..., k, k+1\}$ satisfying $\overline{p}(i) \le \overline{p}(j)$ whenever $i < j$.

Note that $\overline{p}(0) = 0$, $\overline{p}(\tilde{k}+1) = k+1$ and that there are exactly k distinct proper nonzero subspaces among $V_{\overline{p}(1)}, ..., V_{\overline{p}(\tilde{k})}$. Moreover, the surjection $p : \{0, 1, ..., \tilde{k}\} \to \{0, 1, ..., k\}$ satisfies $p(j) = \overline{p}(j)$ whenever $p(j) \neq 0$ and $p^{-1}(0) \cup \{\tilde{k}\} = \overline{p}^{-1}(0) \sqcup \overline{p}^{-1}(k+1)$.

A strict standard extension is a linear embedding, except in the case

$$FlO(m_1, ..., m_k, V) \hookrightarrow FlO(n_1, ..., n_{\tilde{k}}, V')$$

where $\frac{\dim V}{2} - 1$ appeas among $m_1, ..., m_k$ but $\frac{\dim V'}{2} - 1$ does not appear among $n_1, ..., n_{\tilde{k}}$, or $\frac{\dim V}{2}$ appears among $m_1, ..., m_k$ but $\frac{\dim V'}{2} - 1$ or $\frac{\dim V'}{2}$ does not appear among $n_1, ..., n_{\tilde{k}}$.

Of course, in the case of ordinary (i.e. not isotropic) flag varieties, we also need the definition of a *modified standard extension.* By definition, this is a composition $\varphi = d \circ \varphi'$ where

$$\varphi' : \ Fl(m_1, ..., m_k, V) \hookrightarrow Fl(\dim V' - n_{\tilde{k}}, ..., \dim V' - n_1, V'^{\vee})$$

is a strict standard extension and

$$d : \ Fl(\dim V' - n_{\tilde{k}}, ..., \dim V' - n_1, V'^{\vee}) \xrightarrow{\simeq} Fl(n_1, ..., n_{\tilde{k}}, V')$$

is the duality isomorphism. Here $\varphi^*[M_j] = [L_{q(j)}]$ for a map $q : \{0, 1, ..., \tilde{k}\} \to \{0, 1, ..., k\}$ such that $q(0) = 0$, $q(i) \geq q(j)$ whenever $q(i) \neq 0$, $q(j) \neq 0$ and $i \leq j$, and also $q(j) = 0$ implies $j < t$ or $j > t$ for all t with $q(t) \neq 0$.

Example 3.3. (i) Consider the extreme case when $k = 1$ and $\tilde{k}$ is an arbitrary integer greater or equal to 1. Then the surjection $\overline{p} : \{0, 1, ..., \tilde{k}, \tilde{k}+1\} \to \{0, 1, 2\}$ from Definition 3.2,(ii) defines an ordered partition of $\{0, 1, ..., \tilde{k}, \tilde{k}+1\}$ with three parts $\overline{p}^{-1}(0)$, $\overline{p}^{-1}(1)$, $\overline{p}^{-1}(2)$, and a corresponding standard extension

$$G(m, V) \hookrightarrow Fl(m_1, ..., m_{\tilde{k}}, V')$$

has the form

$$(0 \subset V_m \subset V) \mapsto (0 \subset W_1 \subset ... \subset W_s \subset \varepsilon_{s+1}^{-1}(V_m) \subset ... \subset \varepsilon_t^{-1}(V_m) \subset U_{t+1} \subset ... \subset U_{\tilde{k}} \subset V'),$$

where $\{0, 1, ..., s\} = \overline{p}^{-1}(0)$, $\{s+1, ..., t\} = \overline{p}^{-1}(1)$ and $\{t+1, ..., \tilde{k}+1\} = \overline{p}^{-1}(2)$.

(ii) Next, consider the case when $\dim V' = \dim V + 1$. Then $\tilde{k}$ necessarily equals k or $k+1$. Hence, $\dim W_i \leq 1$ and there exists i_0, $0 \leq i_0 \leq k$, such that $W_j = 0$ for $j \leq i_0$ and $\dim W_{i_0+1} = ... = \dim W_{\tilde{k}} = 1$. Consequently, $W_{i_0+1} = ... = W_{\tilde{k}}$. Set $W := W_{i_0+1} = ... = W_{\tilde{k}}$. If $\tilde{k} = k$, then p is a bijection and the corresponding standard extension $\varphi : \ Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_k, V')$ has the form

$$\varphi(0 \subset V_{m_1} \subset ... \subset V_{m_k} \subset V) = \tag{12}$$
$$= \begin{cases} (0 \subset V_{m_1} \oplus W \subset ... \subset V_{m_k} \oplus W \subset V') & for\ i_0 = 0, \\ (0 \subset V_{m_1} \subset ... \subset V_{m_{i_0}} \subset V_{m_{i_0+1}} \oplus W \subset ... \subset V_{m_k} \oplus W \subset V') & for\ 0 < i_0 < k, \\ (0 \subset V_{m_1} \subset ... \subset V_{m_k} \subset V') & for\ i_0 = k. \end{cases}$$

If $\tilde{k} = k+1$, then $p(i_0) = p(i_0+1) = i_0$ and the standard extension $\varphi : \ Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{k+1}, V')$ has the form

$$\varphi(0 \subset V_{m_1} \subset ... \subset V_{m_k} \subset V) = \tag{13}$$
$$= \begin{cases} (0 \subset W \subset V_{m_1} \oplus W \subset ... \subset V_{m_k} \oplus W \subset V') & for\ i_0 = 0, \\ (0 \subset V_{m_1} \subset ... \subset V_{m_{i_0}} \subset V_{m_{i_0+1}} \oplus W \subset ... \subset V_{m_k} \oplus W \subset V') & for\ 0 < i_0 < k, \\ (0 \subset V_{m_1} \subset ... \subset V_{m_k} \subset V_{m_k} \oplus W \subset V') & for\ i_0 = k. \end{cases}$$

(iii) Let $\dim V = 2$ and let $V' = V \oplus V$. Consider the embedding

$$\mathbb{P}(V) = G(1, V) \hookrightarrow Fl(1, 2, 3, V \oplus V), \qquad (0 \subset V_1 \subset V) \mapsto (0 \subset V_1 \subset V \oplus 0 \subset V \oplus V_1 \subset V \oplus V).$$

This embedding is not a standard extension. Here, $\varphi^*[M_1] = \varphi^*[M_3] = [L]$, $\varphi^*[M_2] = 0$. This shows that there is no p as in the definition of strict standard extension, and it is easy to check that φ is also not a modified standard extension.

(iv) Let V' be endowed with non-degenerate symmetric or symplectic form, and $V' = V \oplus \widehat{W}$ where $\widehat{W} = V^{\perp}$ and $\dim \widehat{W} = 2$. Fix an isotropic line $W \subset \widehat{W}$. Then for any increasing sequence $0 < m_1 < ... < m_k \leq [\frac{\dim V}{2}]$ and any s, $1 \leq s \leq k$, there is a standard extension $\varphi : X \to Y$, where $X = FlO(m_1, ..., m_k, V)$ and $Y = FlO(m_1, ..., m_s, m_s+1, ..., m_k+1, V')$, or respectively, $X = FlS(m_1, ..., m_k, V)$ and $Y = FlS(m_1, ..., m_s, m_s+1, ..., m_k+1, V')$. For $s = 0$ there also is a standard extension $\varphi : X \to Y$, where now $Y = FlO(1, m_1+1, ..., m_k+1, V')$

or $Y = FlS(1, m_1+1, ..., m_k+1, V')$, respectively. The embedding φ is given by formula (13) with i_0 substituted by s.

A less canonical, but more intuitive, description of strict standard extensions (respectively, of standard extensions in the isotropic case) is given by the following easily proved proposition.

Proposition 3.4. *Assume that* $\varphi : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$, *respectively,* $\varphi : FlO(m_1, ..., m_k, V) \hookrightarrow FlO(n_1, ..., n_{\tilde{k}}, V')$, *respectively,* $\varphi : FlS(m_1, ..., m_k, V) \hookrightarrow FlS(n_1, ..., n_{\tilde{k}}, V')$ *is a nonconstant strict standard extension corresponding to a surjection* $\overline{p} : \{0, 1, ..., \tilde{k}, \tilde{k}+1\} \to \{0, 1, ..., k, k+1\}$. *Define the flag* $(0 \subset W_1 \subset ... \subset W_{\tilde{k}} \subset V')$ *by setting* $W_i := \ker \varepsilon_i$. *Then there exists a direct sum decomposition*

$$V' = V \oplus \widehat{W} \tag{14}$$

with $\widehat{W} = V^{\perp}$ *in the orthogonal and symplectic case, and such that* $W_i \subset \widehat{W}$, $U_i \supset V$ *for all* i *with* $\varepsilon_i \neq 0$, *and the nonzero operators* $\varepsilon_i : U_i \to V$ *are just projections onto* V *via the decomposition* (14). *Moreover,*

$$\varphi\big(0 \subset V_{\overline{p}(1)} \subset ... \subset V_{\overline{p}(\tilde{k})} \subset V\big) = \big(0 \subset V_{\overline{p}(1)} \oplus W_1 \subset ... \subset V_{\overline{p}(\tilde{k})} \oplus W_{\tilde{k}} \subset V'\big). \tag{15}$$

Lemma 3.5. *In the notation of Proposition 3.4, let* $\underline{w}$ *be a basis of* $\widehat{W}$ *such that all subspaces* W_i *are coordinate subspaces with respect to* $\underline{w}$. *Then, for any splitting* $\widehat{W} = \overline{W} \oplus \overline{\overline{W}}$ *such that* $\overline{W}$ *and* $\overline{\overline{W}}$ *are coordinate spaces, mutually perpendicular within* $\widehat{W}$ *in the orthogonal and symplectic cases, the strict standard extension given by formula* (13) *is the composition of strict standard extensions*

$$Fl(m_1, ..., m_k, V) \hookrightarrow Fl(m'_1, ..., m'_l, V \oplus \overline{W}) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V' = (V \oplus \overline{W}) \oplus \overline{\overline{W}})$$

for which the corresponding flags in $\overline{W}$ *and* $\overline{\overline{W}}$ *are the respective intersections of the flag* $(0 \subset W_1 \subset ... \subset W_{\tilde{k}} \subset W)$ *with* $\overline{W}$ *and* $\overline{\overline{W}}$.

Proof. Direct verification using formula (15). □

4. A sufficient condition for a linear embedding to be a standard extension

In this section we establish our main result concerning linear embeddings of flag varieties. This is a sufficient condition for a linear embedding to be a standard extension.

Consider a flag variety $Fl(m_1, ..., m_k, V)$ and let $\{m_1, ..., m_k\} = R_1 \cup ... \cup R_s$ be a decomposition into a union of s subsets. Denote this decomposition by R. By ordering the elements of R_i we can think of R_i as a type of a flag, and then $Fl(R_i, V)$ is a well-defined flag variety. Moreover, there is a canonical embedding

$$\psi_{R,t_1,...,t_s} : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(R_1, V)^{\times t_1} \times ... \times Fl(R_s, V)^{\times t_s}$$

where by $Fl(R_i, V)^{t_i}$ we denote the direct product of t_i copies of $Fl(R_i, V)$.

If now $\varphi : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$ is an embedding, we say that φ *does not factor through any direct product* if $\varphi \neq \psi \circ \psi_{R,t_1,...,t_s}$ for any decomposition R, any $t_i \in \mathbb{Z}_{\geq 1}$ and any embedding $\psi : Fl(R_1, V)^{\times t_1} \times ... \times Fl(R_s, V)^{\times t_s} \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$. The definition clearly makes sense also in the orthogonal and symplectic cases.

Lemma 4.1. *Let* $\varphi : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$ *be a linear embedding which does not factor through any direct product. Assume that* $\tilde{k} \geq 3$ *and there exist integers* i *and* j, $1 \leq i,\ i+2 \leq j \leq \tilde{k}$, *such that the morphisms* $\pi_i \circ \varphi$ *and* $\pi_j \circ \varphi$ *are not constant maps. Then* *for any l, $i < l < j$, the morphism $\pi_l \circ \varphi$ is not a constant map. Similar statements are true in the orthogonal and symplectic cases.*

Proof. Suppose the contrary, i. e. that there exists l, $i < l < j$, such that the morphism $\pi_l \circ \varphi$ is a constant map, and let $V'_l := \mathrm{im}(\pi_l \circ \varphi) \subset V'$. Then φ induces well-defined embeddings

$$\varphi' : Fl(p(\{0,1,...,l\}), V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V'),$$
$$\varphi'' : Fl(p(\{l,...,\tilde{k}\}), V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V'),$$

where we consider $p(\{0,1,...,l\})$ and $p(\{l,...,\tilde{k}\})$ as types of flags. Moreover, φ clearly factors through the embedding

$$\psi : \ Fl(p(\{0,1,...,l\}), V) \times Fl(p(\{l,...,\tilde{k}\}, V) \to Fl(n_1, ..., n_{\tilde{k}}, V'),$$

where, for $F_1 \in Fl(p(\{0,1,...,l\}), V)$ and $F_2 \in Fl(p(\{l,...,\tilde{k}\}, V)$, the spaces with indices from 1 to l of the flag $\psi(F_1 \times F_2)$ coincide with those of the flag $\varphi'(F_1)$, and the spaces with indices from l to $\tilde{k}$ coincide with those of the flag $\varphi''(F_2)$. The flag $\psi(F_1, F_2)$ is well defined as its space with index l equals V'_l. □

Theorem 4.2. *Let $\varphi : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V')$ be a linear embedding. Assume that all morphisms $\varphi_{p(j),j} : \ G(m_{p(j)}, V) \hookrightarrow G(n_j, V')$ from Proposition 2.3 are strict standard extensions, and that φ does not factor through any direct product. Then φ is a strict standard extension. Analogous statements hold in the orthogonal and symplectic cases.*

Proof. Lemma 4.1 implies that there are s and t, $s < t$, so that $p(j) = 0$ holds precisely for $j \leq s$ and for $j \geq t$.

In the case when there is a single index j such that $\varphi_{p(j),j}$ is a nonconstant morphism, the statement of the theorem is easy. We thus may assume that there are (at least) two indices j and $j+1$, $1 < j < j+1 < t$, so that φ induces nonconstant strict standard extensions

$$\varphi_{p(j),j} : \ G(m_{p(j)}, V) \hookrightarrow G(n_j, V'), \qquad \varphi_{p(j+1),j+1} : \ G(m_{p(j+1)}, V) \hookrightarrow G(n_{j+1}, V').$$

Define subspaces U_j and U_{j+1} of V' by formula (9) in which we put $\varphi = \varphi_{p(j),j}$ and $m = m_{p(j)}$, or $\varphi = \varphi_{p(j+1),j+1}$ and $m = m_{p(j+1)}$, respectively. Let $(0 \subset V_{m_1} \subset ... \subset V_{m_k} \subset V)$ denote an arbitrary point of $Fl(m_1, ..., m_k, V)$. Since by definition

$$\varphi_{p(j),j}(V_{m_{p(j)}}) \subset \varphi_{p(j+1),j+1}(V_{m_{p(j+1)}}) \tag{16}$$

for any subflag $V_{m_{p(j)}} \subset V_{m_{p(j+1)}}$ if $p(j) < p(j+1)$, or for any subflag $V_{m_{p(j+1)}} \subset V_{m_{p(j)}}$ if $p(j+1) > p(j)$, formula (9) implies that U_j is a subspace of U_{j+1}. Next, since the strict standard extensions $\varphi_{p(j),j}$ and $\varphi_{p(j+1),j+1}$ are nonconstant, it follows from Definition 3.1 that there are surjective linear operators $\varepsilon_j : U_j \to V$ and $\varepsilon_{j+1} : U_{j+1} \to V$, such that formula (7) holds for $\varepsilon = \varepsilon_j$, $m = m_{p(j)}$ and $\varepsilon = \varepsilon_{j+1}$, $m = m_{p(j+1)}$, respectively. This, together with (16), means that

$$\varepsilon_j^{-1}(V_{m_{p(j)}}) \subset \varepsilon_{j+1}^{-1}(V_{m_{p(j+1)}}) \tag{17}$$

under the same conditions on $V_{m_{p(j)}}$ and $V_{m_{p(j+1)}}$ as in (16).

Denoting $W_j = \ker \varepsilon_j$ and $W_{j+1} = \ker \varepsilon_{j+1}$, in view of (16) we obtain from (8) that W_j is a subspace of W_{j+1}. The inclusions $U_j \subset U_{j+1}$ and $W_j \subset W_{j+1}$ join into a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\theta_j} & V \\ {\scriptstyle \varepsilon_j}\uparrow & & \uparrow{\scriptstyle \varepsilon_{j+1}} \\ U_j & \hookrightarrow & U_{j+1} \\ \uparrow & & \uparrow \\ W_j & \hookrightarrow & W_{j+1}, \end{array} \tag{18}$$

where θ_j is the induced linear operator. From (17) and (18) we obtain

$$\theta_j(V_{m_{p(j)}}) \subset V_{m_{p(j+1)}}. \tag{19}$$

Now we are going to show that

$$p(j) \ \leq \ p(j+1).$$

Assume the contrary, i.e. $p(j+1) < p(j)$. Then the inclusion (19) implies

$$\theta_j(V_{m_{p(j)}}) \subset \bigcap_{V_{m_{p(j+1)}} \subset V_{m_{p(j)}}} V_{m_{p(j+1)}} = 0.$$

Thus $\theta_j = 0$, and consequently $U_j \subset W_{j+1}$ by diagram (18). This together with formula (7) means that the inclusion (17) extends to a pair of inclusions

$$\varepsilon_j^{-1}(V_{m_{p(j)}}) \subset W_{j+1} \subset \varepsilon_{j+1}^{-1}(V_{m_{p(j+1)}}),$$

for any $(V_{m_{p(j)}}, V_{m_{p(j+1)}}) \in G(m_{p(j)}, V) \times G(m_{p(j+1)}, V)$. Then the exact same argument as in the proof of Lemma 4.1 shows that φ factors through a direct product. Hence the assumption $p(j+1) < p(j)$ is invalid.

Next, we claim that $\theta_j = c_j \mathrm{Id}$ for some nonzero constant c_j. Note that $\theta_j \neq 0$ by the above. Then, since $\varepsilon_j^{-1}(V_{m_{p(j)}}) \subset \varepsilon_{j+1}^{-1}(V_{m_{p(j+1)}})$, we have $\theta_j(V_{m_{p(j)}}) \subset V_{m_{p(j+1)}}$. Taking into account that $\bigcap_{V_{m_{p(j+1)}} \supset V_{m_{p(j)}}} V_{m_{p(j+1)}} = V_{m_{p(j)}}$, we obtain

$$\theta_j(V_{m_{p(j)}}) \subset V_{m_{p(j)}}$$

for any $V_{m_{p(j)}} \in G(m_{p(j)}, V)$. As any 1-dimensional subspace of V is the intersection of all $m_{p(j)}$-dimensional subspaces which contain it, we see that any vector in V is an eigenvector for θ_j. Consequently, we have $\theta_j = c_j \mathrm{Id}$ for $c_j \neq 0$.

The above argument applies to any pair of integers $j, j+1$ where $s+1 < j < t-2$. Therefore, we can construct a commutative diagram

$$\begin{array}{ccccccccc} V & \xrightarrow{\theta_1} & V & \longrightarrow & \cdots & \longrightarrow & V & \xrightarrow{\theta_{\tilde{k}}} & V \\ \uparrow{\scriptstyle \varepsilon_1} & & \uparrow{\scriptstyle \varepsilon_2} & & & & \uparrow{\scriptstyle \varepsilon_{\tilde{k}-1}} & & \uparrow{\scriptstyle \varepsilon_{\tilde{k}}} \\ U_1 & \hookrightarrow & U_2 & \hookrightarrow & \cdots & \hookrightarrow & U_{\tilde{k}-1} & \hookrightarrow & U_{\tilde{k}}, \end{array} \tag{20}$$

where the morphisms ε_i equal zero for $i \leq s$, $i \geq t$, $\theta_i = \mathrm{Id}$ for $i \leq s$ and $i \geq t$, and $\theta_i = c_i \mathrm{Id}$ with $c_i \neq 0$ for $s+1 \leq i \leq t-1$. Here, the spaces $U_1, ..., U_s, U_{t+1}, ..., U_{\tilde{k}}$ are defined as the subspaces of V' which equal the images of the respective constant morphisms $\pi'_1 \circ \varphi, ..., \pi'_s \circ \varphi$, $\pi'_{t+1} \circ \varphi, ..., \pi'_{\tilde{k}} \circ \varphi$, where

$$\pi'_r : \ Fl(n_1, ..., n_{\tilde{k}}, V') \to G(n_r, V')$$

are the natural projections.

Via scaling the morphisms ε_i for $s+1 \leq i \leq t-1$, we can turn the diagram (20) into the diagram (10) in the definition of strict standard extension. An immediate checking shows that our given embedding φ is given by formula (11) for the surjection $\overline{p} : \{0, 1, ..., \tilde{k}, \tilde{k}+1\} \to \{0, 1, ..., k, k+1\}$ where $\overline{p}(j) = p(j)$ for $j \leq t-1$, $\overline{p}(j) = \tilde{k}+1$ for $j \geq t$. □

The next theorem is a more general version of Theorem 4.2.

Theorem 4.3. *If, in the setting of Theorem 4.2, all morphisms $\varphi_{p(j),j}$ are (not necessarily strict) standard extensions, then φ is also a standard extension.*

Proof. First, as in the proof of Theorem 4.2, we assume that there are (at least) two indices j and $j+1$ such that there are nonconstant standard extensions $\varphi_{p(j),j}$ and $\varphi_{p(j+1),j+1}$ as in (4). The reader will easily handle the remaining case.

We will show now that the standard extensions $\varphi_{p(j),j}$ and $\varphi_{p(j+1),j+1}$ are either both strict or are both modified. For this, we need to exclude the following other logical possibilities:
(a) $p(j) \leq p(j+1)$, $\varphi_{p(j),j} : G(m_{p(j)}, V) \hookrightarrow G(n_j, V')$ is a strict standard extension and $\varphi_{p(j+1),j+1} : G(m_{p(j+1)}, V) \hookrightarrow G(n_{j+1}, V')$ is a modified standard extension;
(b) $p(j) > p(j+1)$, $\varphi_{p(j),j}$ is a modified standard extension and $\varphi_{p(j+1),j+1}$ is a strict standard extension;
(c) $p(j) \leq p(j+1)$, $\varphi_{p(j),j}$ is a modified standard extension and $\varphi_{p(j+1),j+1}$ is a strict standard extension;
(d) $p(j) > p(j+1)$, $\varphi_{p(j),j}$ is a strict standard extension and $\varphi_{p(j+1),j+1}$ is a modified standard extension.

(a) Note that the modified standard extension $\varphi_{p(j+1),j+1}$ defines a flag of subspaces $W_{j+1} \subset U_{j+1}$ of V' and a surjective linear operator $\varepsilon_{j+1} : U_{j+1} \to V'^{\vee}$ with $\ker \varepsilon_{j+1} = W_{j+1}$, such that

$$\varphi_{p(j+1),j+1}(V_{m_{p(j+1)}}) = \varepsilon_{j+1}^{-1}((V/V_{m_{p(j+1)}})^{\vee}), \tag{21}$$

where $(V/V_{m_{p(j+1)}})^{\vee}$ is naturally considered as a subspace of $V^{\vee}$. Moreover,

$$W_{j+1} = \bigcap_{V_{m_{p(j+1)}} \subset V} \varphi_{p(j+1),j+1}(V_{m_{p(j+1)}}). \tag{22}$$

Formulas (21) and (22) are corollaries of formulas (7) and (8), respectively.

Now, given $V_{m_{p(j)}} \in G(m_{p(j)}, V)$, we obtain

$$\{0\} = \bigcap_{V_{m_{p(j+1)}} \supset V_{m_{p(j)}}} (V/V_{m_{p(j+1)}})^{\vee}, \tag{23}$$

where the intersection is taken in $(V/V_{m_{p(j)}})^{\vee}$. Using (21)-(23), we find $W_{j+1} = \bigcap_{V_{m_{p(j+1)}} \supset V_{m_{p(j)}}} \varphi_{p(j+1),j+1}(V_{m_{p(j+1)}})$. Therefore,

$$\varphi_{p(j),j}(V_{m_{p(j)}}) \subset W_{j+1} \subset \varphi_{p(j+1),j+1}(V_{m_{p(j+1)}}) \tag{24}$$

for any $V_{m_{p(j+1)}} \in G(m_{p(j+1)}, V)$. In view of (7) and (21), the inclusion (24) coincides with the inclusion (4). Hence, as in the proof of Theorem 4.2, we see that φ factors through a direct product, contrary to our assumption. This contradiction rules out (a).

(b) Given $V_{m_{p(j+1)}} \in G(m_{p(j+1)}, V)$, for any $V_{m_{p(j)}} \subset V_{m_{p(j+1)}}$ we have $\varphi_{p(j+1),j+1}(V_{m_{p(j)}}) \supset \varphi_{p(j),j}(V_{m_{p(j+1)}})$. Hence, there is an inclusion $\varphi_{p(j),j}(V_{m_{p(j+1)}}) \subset \bigcap_{V_{m_{p(j)}} \subset V_{m_{p(j+1)}}} \varphi_{p(j+1),j+1}(V_{m_{p(j)}})$, the right-hand side of which is zero, as it clearly follows from the definition of nonconstant strict standard extension. Thus, $\varphi_{p(j),j}(V_{m_{p(j+1)}}) = \{0\}$ which is a contradiction,since $V_{n_1} \neq 0$.

Cases (c) and (d) are reduced to cases (a) and (b), respectively, via the duality isomorphisms $G(n_j, V') \overset{\simeq}{\to} G(\dim V' - n_j, V'^{\vee})$ and $G(n_{j+1}, V') \overset{\simeq}{\to} G(\dim V' - n_{j+1}, V'^{\vee})$. Thus, all the cases (a)-(d) lead to a contradiction.

The above, together with Lemma 4.1, implies that either all nonconstant morphisms $\varphi_{p(j),j} : G(m_{p(j)}, V) \hookrightarrow G(n_j, V')$ are strict standard extensions, or that they all are modified standard extensions. In the latter case one considers the morphism $d \circ \varphi$, where d is the duality isomorphism. Then by Theorem 4.2, $d \circ \varphi$ is a strict standard extension, and consequently φ is a modified standard extension. □

We now introduce the following condition on a linear embedding

$$\varphi : Fl(m_1, ..., m_k, V) \hookrightarrow Fl(n_1, ..., n_{\tilde{k}}, V),$$

or respectively,

$$\varphi : FlO(m_1, ..., m_k, V) \hookrightarrow FlO(n_1, ..., n_{\tilde{k}}, V)$$

or

$$\varphi : FlS(m_1, ..., m_k, V) \hookrightarrow FlS(n_1, ..., n_{\tilde{k}}, V).$$

(c) *No nonconstant morphism* $\varphi_{p(j),j} : G(m_i, V) \to G(n_j, V')$ *factors through an embedding of a projective subspace into* $G(n_j, V')$*; in the orthogonal and symplectic cases no nonconstant morphism* $\varphi_{p(j),j} : X \to Y$ *for* $X = GO(m_i, V)$ *and* $Y = GO(n_j, V')$*, or* $X = GS(m_i, V)$ *and* $Y = GS(n_j, V')$*, factors through a smooth subvariety of* Y *isomorphic to a grassmannian* $G(m, V'')$ *or a multidimensional quadric in case* $Y = GO(n_j, V')$*; in the case where* $X = GO(s-1, V),\ Y = GO(t-1, V')$ *for* $\dim V = 2s,\ \dim V' = 2t$ *for* $t > s$*, this latter condition should also be imposed on the induced morphism* $\tilde{\varphi}_{p(j),j} : GO(s, V) \to GO(t, V')$.

We say that a linear embedding φ is *admissible* if it does not factor through any direct product and satisfies condition (c).

Our main result in this section is the following.

Corollary 4.4. *An admissible linear embedding* φ *is a standard extension.*

Proof. According to Theorem 4.3, all we need to show is that condition (c) implies that every nonconstant morphism $\varphi_{p(j),j}$ is a standard extension. For usual grassmannians, this follows directly from [3, Thm. 1], which claims that a linear morphism of grassmannians $\varphi_{p(j),j} : X \to Y$ is a standard extension unless it factors through a projective subspace of Y. For isotropic grassmannians, [3, Thm. 1] applies only to the case when $\mathrm{Pic}X \simeq \mathrm{Pic}Y \simeq \mathbb{Z}$, and also implies our claim under this assumption. It remains to consider the situation of a linear morphism $\varphi_{p(j),j} : G(s-1, V) \to G(t-1, V')$ where $\dim V = 2s,\ \dim V' = 2t,\ t \geq s$. In this situation, as stated in Section 2, we always have a commutative diagram

$$\begin{array}{ccc} GO(s-1, V) & \xhookrightarrow{\varphi_{p(j),j}} & GO(t-1, V') \\ \big\downarrow \theta & & \big\downarrow \theta' \\ GO(s, V) & \xhookrightarrow{\tilde{\varphi}_{p(j),j}} & GO(t, V'). \end{array}$$

Here, [3, Thm. 1] applies to the linear morphism $\tilde{\varphi} := \tilde{\varphi}_{p(j),j}$, implying that it is a standard extension whenever it does not factor through a grassmannian or a multidimensional quadric embedded in $GO(t, V')$. Let this standard extension have the form

$$V_s \mapsto V_s \oplus W', \tag{25}$$

where $V' = V \oplus W$ is an orthogonal decomposition and W' is a maximal isotropic subspace of W. We will show that $\varphi := \varphi_{p(j),j}$ is the standard extension

$$V_{s-1} \mapsto V_{s-1} \oplus W'. \tag{26}$$

For this, consider an arbitrary projective line $\mathbb{P}^1$ on $GO(s, V)$, i.e. a smooth rational curve $C \subset GO(s, V)$ such that $\mathcal{O}_{GO(s,V)}(1)|_C \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. It is an exercise to see that there exists an isotropic subspace $W_{\mathbb{P}^1} \subset V$ of dimension $p-2$, such that the restriction $E := \mathcal{S}|_{\mathbb{P}^1}$ of the tautological bundle $\mathcal{S}$ on $GO(s, V)$ is isomorphic to $2\mathcal{O}_{\mathbb{P}^1}(-1) \oplus W_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}$. Hence, by (25), we have

$$E' := \varphi^* \mathcal{S}'|_{\mathbb{P}^1} \simeq 2\mathcal{O}_{\mathbb{P}^1}(-1) \oplus (W_{\mathbb{P}^1} \oplus W') \otimes \mathcal{O}_{\mathbb{P}^1}, \tag{27}$$

where $\mathcal{S}'$ is the tautological bundle on $GO(t-1, V')$.

For any point $x \in \mathbb{P}^1$, consider the projective spaces $\theta^{-1}(x) = \mathbb{P}(E^\vee|_t)$ and $\theta'^{-1}(\tilde{\varphi}(x)) = \mathbb{P}((E')^\vee|_t)$. By definition, $\varphi|_{\theta^{-1}(x)} : \theta^{-1}(x) \to \theta'^{-1}(\tilde{\varphi}(x))$ is a linear embedding of projective spaces, hence it has the form

$$V_{s-1} \mapsto V_{s-1} \oplus W''(x) \tag{28}$$

for some unique isotropic vector subspace $W''(x) \subset V'$. Indeed, $W''(x) = \bigcap_{V_{s-1} \in \theta^{-1}(x)} \varphi(V_{s-1})$ (see (8)). Moreover, by construction, $W'' := \{(x, W''(x))\}_{x \in \mathbb{P}^1}$ is a vector subbundle of E', and the condition that $\varphi^* \mathcal{O}_{GO(t-1,V')}(1) \cong \mathcal{O}_{GO(s-1,V)}(1)$ (see Example 2.2) implies

$$\det W'' \cong \mathcal{O}_{\mathbb{P}^1}. \tag{29}$$

Consider the composition of morphisms of sheaves: $f : W'' \overset{i}{\hookrightarrow} E' \overset{pr}{\to} 2\mathcal{O}_{\mathbb{P}^1}(-1)$ where i is the above mentioned monomorphism and pr is the canonical projection defined by (27). If f is a nonzero morphism, it follows from (29) and Grothendieck's Theorem that W'' contains a direct summand $\mathcal{O}_{\mathbb{P}^1}(a)$ for some $a > 0$. But this contradicts to (27) since i is a monomorphism. Hence, $f = 0$, and by (27), W'' is a subbundle of the trivial bundle $(W_{\mathbb{P}^1} \oplus W') \otimes \mathcal{O}_{\mathbb{P}^1}$. Therefore, in view of (29), W'' is itself a trivial bundle. This means that the space $W''(x)$ does not depend on $x \in \mathbb{P}^1$, but possibly depends only on the choice of projective line $\mathbb{P}^1$. We can set $W''(x) = W''_{\mathbb{P}^1}$. Then

$$W''_{\mathbb{P}^1} \subset W_{\mathbb{P}^1} \oplus W'. \tag{30}$$

Pick a point $x_0 \in \mathbb{P}^1$, so that $W''(x_0) = W''_{\mathbb{P}^1}$. Next, pick another line $\mathbb{P}'^1$ through x_0, distinct from $\mathbb{P}^1$. Then $W''_{\mathbb{P}^1} = W''_{\mathbb{P}'^1}$. Since, as one easily checks, any two points in $GO(s,V)$ can be connected by a chain of projective lines, we conclude that $W''_{\mathbb{P}^1}$ does not depend on the line $\mathbb{P}^1$. We therefore denote this space by W''_0, and the inclusion (30) can be rewritten as

$$W''_0 \subset W_{\mathbb{P}^1} \oplus W', \qquad \mathbb{P}^1 \subset GO(s,V). \tag{31}$$

Now one easily observes that $\bigcap_{\mathbb{P}^1 \subset GO(s,V)} W_{\mathbb{P}^1} = \{0\}$. Hence, (31) implies $W''_0 = \bigcap_{\mathbb{P}^1 \subset GO(s,V)} (W_{\mathbb{P}^1} \oplus W') = W'$. It follows that the linear embedding φ in (28) is $V_{s-1} \mapsto V_{s-1} \oplus W'$, i.e., φ coincides with (26) as claimed. □

Corollary 4.4 provides a sufficient condition, in terms of pure algebraic geometry, for a linear embedding of flag varieties, or varieties of isotropic flags, to be a standard extension.

5. Admissible direct limits of linear embeddings of flag varieties are isomorphic to ind-varieties of generalized flags

We start by recalling the notions of generalized flag and ind-variety of generalized flags introduced in [1, Section 5]. Let V be an arbitrary vector space. A *chain of subspaces in* V is a set $\mathcal{C}$ of pairwise distinct subspaces of V such that for any pair F, $H \in \mathcal{C}$, one has either $F \subset H$ or $H \subset F$. Every chain of subspaces $\mathcal{C}$ is linearly ordered by inclusion. Given a chain $\mathcal{C}$, we denote by $\mathcal{C}'$ (respectively, by $\mathcal{C}''$) the subchain of $\mathcal{C}$ that consists of all subspaces $C \in \mathcal{C}$ which have an immediate successor (respectively, an immediate predecessor) with respect to this ordering.

A *generalized flag in* V is a chain of subspaces $\mathcal{F}$ that satisfies the following conditions:
(i) each $F \in \mathcal{F}$ has an immediate successor or an immediate predecessor, i.e. $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$;
(ii) $V \backslash \{0\} = \cup_{F' \in \mathcal{F}'} F'' \backslash F'$, where $F'' \in \mathcal{F}''$ is the immediate successor of $F' \in \mathcal{F}'$.

In what follows, we assume that V is a countable-dimensional vector space with basis $E = \{e_n\}_{n \in \mathbb{Z}_{>0}}$. A generalized flag $\mathcal{F}$ in V is *compatible with the basis* E if for every $F \in \mathcal{F}$ the set $F \cap E$ is a basis of F. We say that a generalized flag $\mathcal{F}$ is *weakly compatible with* E, if $\mathcal{F}$ is compatible with some basis L of V such that $E \backslash (E \cap L)$ is a finite set.

Example 5.1. Let $V = \mathrm{Span}E$ where $E = \{e_n\}_{n\in\mathbb{Z}_{>0}}$.

(i) Any finite chain $(0 \subset F_1 \subset ... \subset F_k \subset V)$ of coordinate subspaces (i. e. subspaces $F_i \subset V$ satisfying $F_i = \mathrm{Span}\{F_i \cap E\}$ for $1 \leq i \leq k$) is a generalized flag compatible with the basis E. If $\dim F_i < \infty$ for $1 \leq i \leq k$, and if one drops the condition that all F_i are coordinate subspaces, then the chain $(0 \subset F_1 \subset ... \subset F_k \subset V)$ is a generalized flag weakly compatible with E.

(ii) Fix a bijection $\mathbb{Z}_{>0} = \mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$, and let $\prec$ denote the linear order on $\mathbb{Z}_{>0}$, induced by the obvious linear order on $\mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$ in which all elements of $\mathbb{Z}_{<0}$ are larger than all elements of $\mathbb{Z}_{>0}$. Then the chain $\{0, F_j, V\}_{j\in\mathbb{Z}_{>0}}$, where $F_j = \{\mathrm{Span}\{e_i\}_{i\preccurlyeq j}\}$, is a generalized flag compatible with E.

(iii) Fix a bijection $\mathbb{Z}_{>0} = \mathbb{Q}_l \sqcup \mathbb{Q}_r$, where $\mathbb{Q}_l = \mathbb{Q} = \mathbb{Q}_r$, and consider the following linear order on $\mathbb{Q}_l \sqcup \mathbb{Q}_r$: $j \prec t \;\Leftrightarrow\; j \in \mathbb{Q}_l \sqcup \mathbb{Q}_r,\ t \in \mathbb{Q}_l \sqcup \mathbb{Q}_r,\ j < t$, or $j = t,\ j \in \mathbb{Q}_l,\ t \in \mathbb{Q}_r$. Then the chain $\{F'_j, F''_j\}_{j\in\mathbb{Q}_l}$, where $F'_j = \mathrm{Span}\{e_k\}_{k\prec j}$, $F''_j = \mathrm{Span}\{e_k\}_{k\preccurlyeq j}$, is a generalized flag compatible with E.

We define two generalized flags $\mathcal{F}$ and $\mathcal{G}$ in V to be E*–commensurable* if both $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E and there exists an inclusion preserving bijection $\varphi : \mathcal{F} \to \mathcal{G}$ and a finite-dimensional subspace $U \subset V$, such that for every $F \in \mathcal{F}$

$$F \subset \varphi(F) + U, \quad \varphi(F) \subset F + U, \quad \dim(F \cap U) = \dim(\varphi(F) \cap U).$$

Let

$$\mathbf{X} = \mathbf{Fl}(\mathcal{F}, E, V)$$

denote the set of all generalized flags in V that are E-commensurable with $\mathcal{F}$. We now explain that $\mathbf{X}$ has a natural ind-variety structure. Let $V'_n := \mathrm{Span}\{e_j | j \leq n\}$. Then the intersection $\mathcal{F} \cap V'_n$ is a flag in V'_n, and let this flag have type $0 < m'_{n,1} < ... < m'_{n,k_n} < n$ for $k_n \leq n-1$. Since $\dim V'_{n+1} = \dim V'_n + 1 = n+1$, if we set $W'_n := \mathrm{Span}\{e_{n+1}\}$, we have $V'_{n+1} = V'_n \oplus W'_n$ and there is a standard extension $i_n : Fl(m'_{n,1}, ..., m'_{n,k_n}, V'_n) \hookrightarrow Fl(n'_{n+1,1}, ..., n'_{n+1,k_{n+1}}, V'_{n+1})$ given by formulas (12) or (13) in Example 3.4 (where we had no need to use as many subscripts as well as primes).

Note that this standard extension i_n is determined by the two types of flags $(m'_{n,1}, ..., m'_{n,k_n})$ and $(n'_{n+1,1}, .., n'_{n+1,k_{n+1}})$, and by the choice of W'_{n+1}. In [1] it is shown that $\mathbf{Fl}(\mathcal{F}, E, V)$ is naturally identified with the direct limit

$$\varinjlim Fl(m'_{n,1}, ..., m'_{n,k_n}, V'_n)$$

of the embeddings i_n. In particular, this equips $\mathbf{Fl}(\mathcal{F}, E, V)$ with the structure of an ind-variety.

Let's now consider the case when V is endowed a nondegenerate symmetric or symplectic bilinear form $(\ ,\ )$. Here we assume that either the basis E is isotropic and is enumerated as $\{e_n, e^n\}_{n\in\mathbb{Z}_{>0}}$ where $(e_n, e^n) = 1$ for $n \in \mathbb{Z}_{>0}$, or that E is enumerated as $\{e_n, e_0, e^n\}_{n\in\mathbb{Z}_{>0}}$ where e_n and e^n are isotropic vectors satisfying $(e_n, e^n) = 1$ for $n \in \mathbb{Z}_{>0}$ and e_0 satisfies $(e_0, e_n) = (e_0, e^n) = 0$, $(e_0, e_0) = 1$. This latter enumeration of E is possible only in the case of a symmetric form. We define a generalized flag $\mathcal{F}$ to be *isotropic* if it consists of isotropic and coisotropic subspaces (a subspace F is *coisotropic* if $F^\perp$ is isotropic) and is invariant under taking orthogonal complement. In the current case, where $\dim V = \infty$, this definition is more convenient for our purposes than the consideration of "purely isotropic" flags as in Sections 2, 3 and 4. Note that an isotropic generalized flag is determined by its subchain of isotropic spaces.

Example 5.2. Consider the case where V is endowed with a nondegenerate symmetric form and the basis of V is enumerated as $\{e_n, e_0, e^n\}_{n\in\mathbb{Z}_{>0}}$ as above. Set $F^l_j = \mathrm{Span}\{e_n\}_{n>j, j\geq 0}$, $F^r_j = (F^l_j)^\perp$. Then $F^l_j \supset F^l_k$, $F^r_j \subset F^r_k$, $F^l_j \subset F^r_k$ for $k \geq j$, and $\{F^l_j, F^r_j\}_{j\in\mathbb{Z}_{\geq 0}}$ is a maximal isotropic generalized flag compatible with E.

By $\mathbf{FlO}(\mathcal{F}, E, V)$, or respectively $\mathbf{FlS}(\mathcal{F}, E, V)$, we denote the set of all generalized flags which are E-commensurable with a fixed isotropic flag $\mathcal{F}$ compatible with E. To define an ind-variety structure on $\mathbf{FlO}(\mathcal{F}, E, V)$ or $\mathbf{FlS}(\mathcal{F}, E, V)$, set $V'_n = \mathrm{Span}\{e_j, e^j\}_{j\leq n}$ or respectively $V'_n = \mathrm{Span}\{e_j, e_0, e^j\}_{j\leq n}$. Then $\mathcal{F} \cap V'_n$ has an isotropic subflag of type $0 < m'_{n,1} < ... < m'_{n,k_n} \leq [\frac{n}{2}]$, and there is a standard extension

$$\psi_n : \ FlO(m'_{n,1}, ..., m'_{n,k_n}, V'_n) \hookrightarrow FlO(m'_{n+1,1}, ..., m'_{n+1,k_{n+1}}, V'_n)$$

or

$$\psi_n : \ FlS(m'_{n,1}, ..., m'_{n,k_n}, V'_n) \hookrightarrow FlS(m'_{n+1,1}, ..., m'_{n+1,k_{n+1}}, V'_{n+1}),$$

determined uniquely by the isotropic 1-dimensional subspace $W_n = \mathrm{Span}\{e_{n+1}\}$. One can show that the direct limit of the embeddings ψ_n is identified with $\mathbf{FlO}(\mathcal{F}, E, V)$, or respectively $\mathbf{FlS}(\mathcal{F}, E, V)$, and hence $\mathbf{FlO}(\mathcal{F}, E, V)$ and $\mathbf{FlS}(\mathcal{F}, E, V)$ are ind-varieties [1].

Next, we will relate an arbitrary direct limit of strict standard extensions to the ind-varieties $\mathbf{Fl}(\mathcal{F}, E, V)$, $\mathbf{FlO}(\mathcal{F}, E, V)$, or $\mathbf{FlS}(\mathcal{F}, E, V)$. First, consider a chain of strict standard extensions

$$\varphi_N : Fl(m_{N,1}, ..., m_{N,k_N}, V_N) \hookrightarrow Fl(m_{N+1,1}, ..., m_{N+1,k_{N+1}}, V_{N+1}) \tag{32}$$

for some choice of vector spaces V_N, $\dim V_{N+1} > \dim V_N$ for $N \in \mathbb{Z}_{>0}$. Then, according to Proposition 3.4, we may choose vector spaces $\widehat{W}_N$, together with isomorphisms

$$V_{N+1} = V_N \oplus \widehat{W}_N,$$

and flags in $\widehat{W}_N$

$$W_{N,1} \subset ... \subset W_{N,k_N} \subset \widehat{W}_N,$$

such that each φ_N is given by:

$$\varphi_N(0 \subset V_{m_{N,1}} \subset ... \subset V_{m_{N,k_N}} \subset V_N) = (0 \subset V_{m_{N,1}} \oplus W_{N,1} \subset ... \subset V_{m_{N,k_N}} \oplus W_{N,k_N} \subset V_{N+1}).$$

Set

$$V := \lim_{\to} V_N.$$

Our aim is to define a basis E of V and a generalized flag $\underline{\mathcal{F}}$ compatible with E, so that the direct limit of the strict standard extensions φ_N can be identified with $\mathbf{Fl}(\underline{\mathcal{F}}, E, V)$. Fix a flag $F_1 = (0 \subset V_{1,1} \subset ... \subset V_{1,k_1} \subset V_1) \in Fl(m_{1,1}, ..., m_{1,k_1}, V_1)$. Choose a basis

$$E = \{e_\alpha\}_{\alpha \in \mathbb{Z}_{>0}}$$

of V such that, for all subspaces T of V of the form $V_{1,1}, ..., V_{1,k_1}$ and $W_{N,j}$ for N and j, the set $T \cap E$ is a basis of T. Consider the following equivalence relation $\sim$ on the set E. We write

$$e_\alpha \sim e_{\tilde{\alpha}}$$

if there exists $N_\alpha \in \mathbb{Z}_{>0}$ such that, for any $N \geq N_\alpha$, there is no space of the flag $\varphi_N \circ \varphi_{N-1} \circ ... \circ \varphi_1(F_1)$ containing e_α but not $e_{\tilde{\alpha}}$, or vice versa. Using the fact that all embeddings φ_N are strict standard extensions, one checks that $\sim$ is an equivalence relation. Denote by $[e_\alpha]$ the equivalence class of the vector e_α.

Next, we claim that, by construction, the set A of equivalence classes $[e_\alpha]$ is linearly ordered, and we will denote this linear ordering by the symbol $\prec$. Indeed, let $[e_\alpha] \neq [e_\beta]$. For $n \geq \max\{N_\alpha, N_\beta\}$, consider the flag $\varphi_N \circ \varphi_{N-1} \circ ... \circ \varphi_1(F_1)$ and take its smallest subspaces containing respectively e_α and e_β. Since $[e_\alpha] \neq [e_\beta]$, it follows that these spaces are not equal. By definition, we have $[e_\alpha] \prec [e_\beta]$ if the smallest space of the flag $\varphi_N \circ \varphi_{N-1} \circ ... \circ \varphi_1(F_1)$ containing e_α is smaller than the smallest space of the same flag containing e_β.

Finally, we define a generalized flag $\underline{\mathcal{F}}$, compatible with the basis E, and determined by the above order on E. For this, we associate two subspaces of V to any equivalence class $a = [e_\alpha]$:

$$F'_a = \mathrm{Span}\{e_\beta \mid [e_\beta] \prec a\}, \qquad F''_a = \mathrm{Span}\{e_\beta \mid [e_\beta] \preccurlyeq a\}. \tag{33}$$

Then the set of vector subspaces of V

$$\underline{\mathcal{F}} = \{F'_a, F''_a\}_{a\in A} \tag{34}$$

is easily seen to be a generalized flag in V compatible with E.

If, instead of (32), we consider standard extensions

$$\psi_N : \; FlO(m_{N,1}, ..., m_{N,k_N}, V_N) \hookrightarrow FlO(m_{N+1,1}, ..., m_{N+1,k_{N+1}}, V_{N+1}) \tag{35}$$

or

$$\psi_N : \; FlS(m_{N,1}, ..., m_{N,k_N}, V_N) \hookrightarrow FlS(m_{N+1,1}, ..., m_{N+1,k_{N+1}}, V_{N+1}), \tag{36}$$

a similar construction of a relevant basis E goes through. First of all, in the case of (35), for our purposes it suffices to assume that that the dimension of all spaces V_N are simultaneously odd or even. We require E to have the form $\{e_n, e_0, e^n\}_{n\in\mathbb{Z}_{>0}}$ in the odd case, and the form $\{e_n, e^n\}_{n\in\mathbb{Z}_{>0}}$ in the even case. This latter form applies also to the case of (36). In all cases, E has to be chosen by the same condition that all subspaces of the form $V_{1,1}, ..., V_{1,k_1}$ and W_{N,k_j} for $N \in \mathbb{Z}_{>0}$ are generated by subsets of E. Next, in order to define a linear order on E, one applies to the vectors e_n the procedure outlined above, and then sets $e^k \prec e^l \;\Leftrightarrow\; e_l \prec e_k$. Finally, whenever there is a vector e_0 one puts $e_n \prec e_0 \prec e^k$ for any $k, n \in \mathbb{Z}_{>0}$. Then the generalized flag $\underline{\mathcal{F}}$ determined by formulas (33) and (34) is isotropic (in the sense of the definition of the beginning of this section) and an ind-variety $\mathbf{FlO}(\underline{\mathcal{F}}, E, V)$, or respectively $\mathbf{FlS}(\underline{\mathcal{F}}, E, V)$ is well defined.

We are now ready for the following theorem.

Theorem 5.3. *There is an isomorphism of ind-varieties*

$$\varinjlim Fl(m_{N,1}, ..., m_{N,k_N}, V_N) \simeq \mathbf{Fl}(\underline{\mathcal{F}}, E, V).$$

Similarly, in the orthogonal and symplectic cases, there are isomorphisms of ind-varieties

$$\varinjlim FlO(m_{N,1}, ..., m_{N,k_N}, V_N) \simeq \mathbf{Fl}(\underline{\mathcal{F}}, E, V),$$
$$\varinjlim FlS(m_{N,1}, ..., m_{N,k_N}, V_N) \simeq \mathbf{Fl}(\underline{\mathcal{F}}, E, V).$$

Proof. We consider only the case of ordinary flag varieties, and leave the other cases to the reader. Note first that $(m_{N,1}, ..., m_{N,k_N})$ is the type of the flag $\underline{\mathcal{F}} \cap V_N$, so that $\mathbf{Fl}(\underline{\mathcal{F}}, E, V) = \varinjlim Fl(m_{N,1}, ..., m_{N,k_N}, V_N)$ where the direct limit is taken with respect to the embeddings

$$i_{\dim V_{N+1}-1} \circ ... \circ i_{\dim V_N} : \; Fl(m_{N,1}, ..., m_{N,k_N}, V_N) \hookrightarrow Fl(m_{N+1,1}, ..., m_{N+1,k_{N+1}}, V_{N+1}).$$

The embeddings i_n were introduced in the first part of this section, and are given by formulas (12) and (13), respectively.

However, we claim that our fixed standard extension φ_N equals the composition $i_{\dim V_{N+1}-1} \circ ... \circ i_{\dim V_N}$. This follows from an iterated application of Lemma 3.5 to the decompositions

$$V_{N+1} = V'_{\dim V_{N+1}-1} \oplus \mathrm{Span}\{e_{\dim V_{N+1}}\},$$
$$V'_{\dim V_{N+1}-1} = V'_{\dim V_{N+1}-2} \oplus \mathrm{Span}\{e_{\dim V_{N+1}-1}\}, ...,$$
$$V'_{\dim V_N+1} = V_N \oplus \mathrm{Span}\{e_{\dim V_N+1}\},$$

and from the observation that the corresponding standard extensions

$$Fl(m_{n,1}, ..., m_{n,k_n}, V'_n) \hookrightarrow Fl(m_{n+1,1}, ..., m_{n+1,k_{n+1}}, V'_{n+1})$$

arising in this way, are determined simply by the splitting $V'_{n+1} = V'_n \oplus \mathrm{Span}\{e_{n+1}\}$. Since the standard extension i_n is determined by the same decomposition, the statement follows. □

The following corollary can be considered as the main result of this paper.

Corollary 5.4. *The direct limit of any admissible sequence of linear embeddings,* $\varinjlim Fl(m_{N,1}, ...m_{N,k_N}, V_N)$, $\varinjlim FlO(m_{N,1}, ...m_{N,k_N}, V_N)$, *or* $\varinjlim FlS(m_{N,1}, ...m_{N,k_N}, V_N)$, *is a homogeneous ind-variety for the group* $SL(\infty)$, $O(\infty)$ *or* $Sp(\infty)$, *respectively.*

The claim of Corollary 5.4 can be derived more directly from Corollary 4.4 by showing that any direct limit of standard extensions is a homogeneous ind-variety, but Theorem 5.3 provides an explicit description of such a direct limit as an appropriate ind-variety of generalized flags. We should also point out that homogeneous ind-varieties of the ind-groups $GL(\infty)$, $SL(\infty)$, $O(\infty)$, $Sp(\infty)$ have been studied in papers preceding [1], see [2] and the references therein.

6. Appendix

In this appendix, we construct ind-varieties which are not isomorphic to ind-varieties of generalized flags, but nevertheless are direct limits of linear embeddings of flag varieties. Here we use the notation $\mathbb{P}(V)$ also for a countable-dimensional vector space. $\mathbb{P}(V)$ is the ind-variety of 1-dimensional subspaces of V. We also write $\mathbb{P}^\infty$ instead of $\mathbb{P}(V)$ when we do not need to specify V.

First, consider the following chain of linear embeddings

$$... \hookrightarrow Fl(1, 2^n - 1, V_n) \overset{k_n}{\hookrightarrow} G(1, V_n) \times G(2^n - 1, V_n) \overset{j_n}{\hookrightarrow} Fl(1, 2^{n+1} - 1, V_n \oplus V_n) \overset{k_{n+1}}{\hookrightarrow}$$

$$\overset{k_{n+1}}{\hookrightarrow} G(1, V_n \oplus V_n) \times G(2^{n+1} - 1, V_n \oplus V_n) \hookrightarrow ... ,$$

where $\dim V_n = 2^n$, k_n and k_{n+1} are the canonical embeddings, and $j_n(V_1, V_{2^n-1}) = (V_1 \subset V \oplus 0 \subset V \oplus V_{2^n-1})$ for subspaces $V_1,\ V_{2^n-1} \subset V$ of respective dimensions 1 and $2^n - 1$. Clearly, the embedding

$$j_n \circ k_n : Fl(1, 2^n - 1, V_n) \hookrightarrow Fl(1, 2^{n+1} - 1, V_n)$$

is linear but does not satisfy condition (b) of Theorem 4.3 as it factors through the embedding k_n. The direct limit $\varinjlim Fl(1, 2^n - 1, V_n)$ is isomorphic as an ind-variety to the direct limit of embeddings

$$G(1, V_n) \times G(2^n - 1, V_n) \overset{k_{n+1} \circ j_n}{\hookrightarrow} G(1, V_n \oplus V_n) \times G(2^n - 1, V_n \oplus V_n),$$

which is easily checked to be isomorphic to the direct product $\mathbb{P}(V) \times \mathbb{P}(V)$ for a countable-dimensional vector space V. The ind-variety $\mathbb{P}(V) \times \mathbb{P}(V)$ is not isomorphic to an ind-variety of generalized flags.

Next, we will give a more interesting example in which condition (c) is not satisfied. More precisely, we will construct a linear embedding $\varphi : Fl(m_1, m_2, V) \hookrightarrow Fl(n_1, n_2, V')$ that will have the property that $p(1) = 1$, $p(2) = 2$, $\varphi_{2,2} : G(m_2, V) \to G(n_2, V')$ is a standard extension, but $\varphi_{1,1} : G(m_1, V) \to G(n_1, V')$ factors through a projective subspace of $G(n_1, V')$.

Let $3 \leq \dim V < \infty$, fix positive integers $m_1,\ m_2,\ 1 < m_1 < m_2 < \dim V$, and let V^0 be a subspace of V of dimension $\dim V - m_1 + 1$. Consider the rational morphism

$$\gamma : G(m_1, V) \dashrightarrow \mathbb{P}(V^0),\ V_{m_1} \mapsto V_{m_1} \cap V^0.$$

Assume $G(m_1, V)$ is embedded into $\mathbb{P}(\wedge^{m_1} V)$ via the Plücker embedding, and let $Y := \{V_{m_1} \in G(m_1, V) \mid \dim(V_{m_1} \cap V^0) \geq 2\}$. A standard computation in linear algebra shows that
(i) there exists a subspace $W \subset \wedge^{m_1} V$ of codimension $\dim V - m_1 + 1$, such that $Y = G(m_1, V) \cap \mathbb{P}(W)$;
(ii) there is an isomorphism $g : (\wedge^{m_1} V)/W \overset{\simeq}{\to} V^0$ satisfying

$$\gamma(V_{m_1}) = g(\wedge^{m_1} V_{m_1} + W) \tag{37}$$

(in particular, this implies that γ is regular on $G(m_1, V) \setminus Y$);
(iii) there exists a vector space U containing $\wedge^{m_1} V$ as a subspace, together with a surjective operator $\varepsilon : U \twoheadrightarrow V$ with $\ker \varepsilon = W$.

In addition, we may suppose that m_1 is large enough so that there exists a subspace Z of W such that the morphism $\varphi' :\ G(m_1, V) \to \mathbb{P}((\wedge^{m_1} V)/Z),\ V_{m_1} \mapsto \wedge^{m_1} V_{m_1} + Z$ is an embedding. Set $V' := U$, $n_1 = \dim Z + 1$, $n_2 = \dim W + m_2$. The inclusion $\wedge^{m_1} V \subset V'$ yields an embedding $j : \mathbb{P}((\wedge^{m_1} V)/Z) \hookrightarrow G(n_1, V'),\ v + Z \mapsto \mathrm{Span}\{v + Z\}$. Define $\varphi_{1,1} : G(m_1, V) \to G(n_1, V')$ as the composition $j \circ \varphi'$, and let $\varphi_{2,2} : G(m_2, V) \to G(n_2, V')$ be the standard extension defined by the flag $(W \subset U)$.

We show now that, given a flag $(0 \subset V_{m_1} \subset V_{m_2} \subset V)$, one has $\varphi_{1,1}(V_{m_1}) \subset \varphi_{2,2}(V_{m_2})$, and hence there is a well-defined embedding

$$\varphi :\ Fl(m_1, m_2, V) \hookrightarrow Fl(n_1, n_2, V'),\ (V_{m_1} \subset V_{m_2}) \mapsto (\varphi_{1,1}(V_{m_1}) \subset \varphi_{2,2}(V_{m_2})).$$

Indeed, in view of (37), the rational morphism γ decomposes as

$$\gamma :\ G(m_1, V) \overset{\varphi'}{\hookrightarrow} \mathbb{P}((\wedge^{m_1} V)/Z) \overset{q}{\dashrightarrow} \mathbb{P}((\wedge^{m_1} V)/W) \underset{\simeq}{\overset{g}{\to}} \mathbb{P}(V^0),$$

$$V_{m_1} \overset{\varphi'}{\mapsto} \wedge^{m_1} V_{m_1} + Z \overset{q}{\mapsto} \wedge^{m_1} V_{m_1} + W \overset{g}{\mapsto} V_{m_1} \cap V^0,$$

where q is a rational surjective morphism. If $V_{m_1} \cap V^0 =: V_1$ is a 1-dimensional space, i.e. if q is regular at the point $\wedge^{m_1} V_{m_1} + Z \in \mathbb{P}((\wedge^{m_1} V)/Z)$, then the inclusion $V_{m_1} \subset V_{m_2}$ implies $V_1 \subset V_{m_2}$. Hence, $\varphi_{1,1}(V_{m_1}) = \wedge^{m_1} V_{m_1} + Z \subset \wedge^{m_1} V_{m_1} + W = \varepsilon^{-1}(V_1) \subset \varepsilon^{-1}(V_{m_2}) = \varphi_{2,2}(V_{m_2})$. In the remaining case when $\dim(V_{m_1} \cap V^0) \geq 2$, we have $\wedge^{m_1} V_{m_1} \subset W$ by property (i), and therefore $\varphi_{1,1}(V_{m_1}) = \wedge^{m_1} V_{m_1} + Z \subset W \subset \varepsilon^{-1}(V_{m_2}) = \varphi_{2,2}(V_{m_2})$.

Finally, we have the following proposition.

Proposition 6.1. *Let $\{\varphi_k :\ Fl(m_{k,1}, m_{k,2}, V_k) \to Fl(m_{k+1,1}, m_{k+1,2}, V_{k+1})\}_{k \geq 1}$ be a chain of embeddings as constructed above. The ind-variety $\mathbf{X}$ obtained as the direct limit of this chain is not isomorphic to an ind-variety of generalized flags.*

Proof. Assume to the contrary that $\mathbf{X}$ is isomorphic to $\mathbf{Y}$ for some ind-variety of generalized flags $\mathbf{Y}$. Since the embeddings φ_k are linear, it follows that $\mathrm{Pic}\mathbf{X} \simeq \mathbb{Z} \times \mathbb{Z}$. Therefore $\mathrm{Pic}\mathbf{Y} \simeq \mathbb{Z} \times \mathbb{Z}$, and consequently, $\mathbf{Y}$ is isomorphic to $\mathbf{Fl}(F', E', V')$ for some countable-dimensional vector space V', some basis E' of V', and some flag $F' = (F'_1 \subset F'_2)$ in V' of length 2. Since the morphisms $(\varphi_k)_{1,1} :\ G(m_{k,1}, V_k) \to G(m_{k+1,1}, V_{k+1})$ factor through projective spaces, the ind-variety $\mathbf{X}$ projects onto $\mathbb{P}^\infty$ in a way that the line bundle $\mathcal{O}_{\mathbf{X}}(1, 0)$ is trivial along the fibers of the projection. Therefore, we infer that $\dim F'_1 = 1$ or $\mathrm{codim}_{V'} F'_2 = 1$. This follows from the fact that the ind-variety $\mathbb{P}^\infty$ is not isomorphic to any ind-grassmannian $\mathbf{Fl}(F, E', V')$, where F is a single subspace with $\dim F \geq 2$ and $\mathrm{codim}_{V'} F' \neq 1$, see [3, Thm. 2]. Consequently, the flag $F' = (F'_1 \subset F'_2)$ can be chosen with $\dim F'_1 = 1$ (in the case where $\mathrm{codim}_{V'} F'_2 = 1$ one replaces V' by its restricted dual space defined by the basis E').

The standard extensions $(\varphi_k)_{2,2} :\ G(m_{k,2}, V_k) \to G(m_{k+1,2}, V_{k+1})$ allow to identify $\varinjlim G(m_{k,2}, V_k)$ with an ind-grassmannian $\mathbf{Fl}(F_\infty, E, V)$, where F_∞ is a subspace of $V = \varinjlim V_k$ and E is an appropriate basis of V. Moreover, we have $\dim F_\infty = \infty = \mathrm{codim}_V F_\infty$, as the construction of φ_k shows that $\lim\limits_{k \to \infty} m_{k,2} = \infty = \lim\limits_{k \to \infty} (\dim V_k - m_{k,2})$. After identifying the triples (F_∞, E, V) and (F'_2, E', V'), we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{X} & \overset{\sigma}{\underset{\sim}{\longleftarrow}} & \mathbf{Fl}(F, E, V) \\ & \searrow^{\pi_{\mathbf{X}}} & \downarrow \pi \\ & & \mathbf{Fl}(F_\infty, E, V), \end{array}$$

where π is the natural projection and σ is an isomorphism of ind-varieties. The fibers of both projections $\pi_{\mathbf{X}}$ and π are isomorphic to $\mathbb{P}^\infty$.

We will show now that the existence of the isomorphism $\mathbf{X} \overset{\sigma}{\underset{\sim}{\leftarrow}} \mathbf{Fl}(F, E, V)$ is contradictory. Recall that the group $GL(E, V)$ of invertible finitary linear operators defined by E (i.e. the group of invertible linear generators on V each of which fixes all but finitely many elements of E) acts on $\mathbf{Fl}(F, E, V)$ and $\mathbf{Fl}(F_\infty, E, V)$, and the line bundle $\mathcal{O}(1,0) := \sigma^*\mathcal{O}_{\mathbf{X}}(1,0)$ on $\mathbf{Fl}(F, E, V)$ admits a $GL(E, V)$-linearization. This linearization is unique when restricted to $SL(E, V)$. If we compute the $SL(E, V)$-module $\Gamma := H^0(\mathbf{Fl}(F, E, V), \mathcal{O}(1, 0))$, we see that $\Gamma \simeq \varprojlim H^0(\pi_{k*}(\mathcal{O}(1,0)|_{Fl(1,m_{k,2},V_k)}))$, where here $\pi_k : Fl(1, m_{k,2}, V_k) \to Gr(m_{k,2}, V_k)$ denote the natural projections. Consequently,

$$\Gamma \simeq \varprojlim V_k^* \simeq V^*.$$

On the other hand, since σ^* induces an $SL(E, V)$-linearization on $\mathcal{O}_{\mathbf{X}}(1, 0)$, and consequently an isomorphism of $SL(E, V)$-modules $\Gamma \xrightarrow{\sim} H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(1, 0))$, we can compute Γ via the system of projections $\tau_k : Fl(m_{k,1}, m_{k,2}, V_k) \to G(m_{k,2}, V_k)$. This yields

$$\Gamma \simeq \varprojlim H^0(\tau_{k*}(\mathcal{O}_{\mathbf{X}}(1,0)|_{Fl(m_{k,1},m_{k,2},V_k)})) \simeq \varprojlim \wedge^{m_k} V_k^*.$$

However, $\varprojlim \wedge^{m_k} V_k^*$ is not isomorphic to V^* as an $SL(E, V)$-module. To see this, it is enough to observe that $\varprojlim \wedge^{m_k} V_k^*$ and V^* are non-isomorphic after restriction to $SL(V_k)$ for large k. We have a contradiction as desired. □

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